AHD AKADEMİ JOURNAL

Volume 1 Issue 1

2026 March

AHD AKADEMI DERGISI

Cilt 1 • Sayı 1 • 2026

ISSN: 1327-1327

İÇİNDEKİLER

https://www.ahdakademi.com

Ahd Akademi Dergisi, 1(1): 1–5
DOI:10.1327/1327
Received: 01 01 2026
Revised: 10 03 2026
Accepted: 15 03 2026
Published: 21 03 2026

Research Article

The Unique Midpoint Property in Ultrametric Spaces

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Abstract:
In this paper we characterize ultrametric spaces that satisfy the unique midpoint property and conclude that such a space must consist of a single point or three points.

Keywords: ultrametric,unique midpoint property, topology
Mathematics Subject Classification: Primary 34B24, 47A70; Secondary 34L20, 47B25

1. Introduction

Let X be a set and $d : X \times X \to \mathbb{R}$ be a function. If the function d satisfies the following properties for all $x, y, z \in X$:

$(d1)$ $d(x, y) \geq 0$
$(d2)$ $d(x, y) = 0$ iff $x = y$
$(d3)$ $d(x, y) = d(y, x)$
$(d4)$ $\max\{d(x, y), d(y, z)\} \geq d(x, z)$

then d is called an ultrametric function on X, and the pair (X, d) is called ultrametric space.It is obvious that every ultrametric space is a metric space. There are also interesting properties that balls provide. Here are two important properties that we use throughout this article: Let (X, d) be an ultrametric space, $x, y \in X$ and $r_1, r_2 > 0$. If $B(x, r_1) \cap B(y, r_2) \neq \emptyset$, then $B(x, r_1) \subseteq B(y, r_2)$ or $B(y, r_2) \subseteq B(x, r_1)$. And also
If $y \in B(x, r_1)$, then $B(x, r_1) = B(y, r_1)$. Let (X, d) be a metric spaces. If for an arbitrary pair $x, y \in X$ there exists a unique $z \in X$ such that $d(x, z) = d(z, y)$, then the point z is called the midpoint of x and y. This space is said to have the unique midpoint property (UMP).

Since the topic of this article is the existence of the unique midpoint property in ultrametric spaces, no extra information on either the ultrametric space or the unique midpoint property is presented, but for convenience, the topic of the ultrametric space and the unique midpoint property can be followed through the articles [1], [2] , [3], and [4].

2. Assumption: There exist an ultrametric space with the unique midpoint property

Let (X, d) be an ultrametric space with the unique midpoint property. For any given points $x, m_0 \in X$, by our assumption, there exists a unique point $m_1 \in X$ such that

$$d(x, m_1) = d(m_1, m_0)$$

and by the ultrametric axiom $(d4)$, it is obtained that

$$d(x, m_1) = d(m_1, m_0) \geq d(x, m_0)$$

.

In this situation there are two cases; (case 1) $d(x, m_1) = d(m_1, m_0) > d(x, m_0)$ and (case 2)$d(x, m_1) = d(m_1, m_0) = d(x, m_0)$.

Case 1: $d(x, m_1) = d(m_1, m_0) > d(x, m_0)$

Let $m_2 \in X$ be the unique midpoint of x and m_1 so

$$d(x, m_2) = d(m_1, m_2)$$

Then by ultrametric axiom $(d4)$

$$max\{d(x, m_2), d(m_2, m_1)\} \geq d(x, m_1)$$

must be holds. by the case

$$d(m_1, m_2) = d(x, m_2) \geq d(x, m_1) > d(x, m_0)$$

and

$$d(m_1, m_2) = d(x, m_2) \geq d(m_0, m_1) > d(x, m_0)$$

So there are two subcases:

Subcase 1:

$$d(m_1, m_2) = d(x, m_2) = d(x, m_1) = d(m_0, m_1) > d(x, m_0)$$

By the ultrametric axiom:

$$\max\{d(m_0, m_2), d(x, m_0)\} \geq d(x, m_2)$$

By the Case 1:

$$\max\{d(m_0, m_2), d(x, m_0)\} > d(x, m_2)$$

Must hold. Therefore, the following equality is imperative:

$$\max\{d(m_0, m_2), d(x, m_0)\} = d(m_0, m_2)$$

Hence

$$d(m_0, m_2) > d(x, m_0)$$

Also:

$$\max\{d(x, m_0), d(m_2, x)\} \geq d(m_0, m_2) > d(x, m_0)$$

$$d(x, m_2) \geq d(m_0, m_2) > d(x, m_0)$$

By the uniqueness of the midpoint of x and m_0:

$$d(x, m_2) > d(m_0, m_2) > d(x, m_0)$$

Therefore:

$$\max\{d(m_0, m_2), d(x, m_0)\} \geq d(x, m_2)$$

$$d(x, m_2) > d(m_0, m_2) \geq d(x, m_2)$$

This is a contradiction.
Subcase 2: $d(m_1, m_2) = d(x, m_2) > d(m_0, m_1) = d(x, m_1) > d(x, m_0)$
By the ultrametric property:

$$\max\{d(x, m_2), d(x, m_0)\} \geq d(m_0, m_2)$$

$$d(x, m_2) \geq d(m_0, m_2)$$

By the uniqueness of the midpoint of x and m_0:

$$d(x, m_2) > d(m_0, m_2)$$

Also by the ultrametric property:

$$\max\{d(m_0, m_2), d(m_0, m_1)\} \geq d(m_1, m_2)$$

If $d(m_0, m_2) \geq d(m_1, m_2)$ then we get:

$$d(x, m_2) > d(m_0, m_2) \geq d(m_1, m_2) = d(x, m_2)$$

which is a contradiction.
If

$$d(m_0, m_1) \geq d(m_1, m_2)$$

then we get:

$$d(m_1, m_2) > d(m_0, m_1) \geq d(m_1, m_2)$$

which is another contradiction. Thus, it is shown that Case 1 never occurs. Hence Case 2 must be satisfied.

Theorem 1. *Let $B(x, r)$ be an open ball and $x, m_0 \in B(x, r)$. If the midpoint, say m_1, of x and m_0 is an element of $B(x, r)$, then $B(x, r)$ cannot contain any element other than x, m_0 and m_1.*

Proof. Assume that there exists $z \in B(x, r)$ such that $z \neq \{x, m_0, m_1\}$. It is clear that

$$d(x, m_0) = d(m_0, m_1) = d(m_1, x)$$

since Case 2 holds. Moreover, by the uniqueness of the midpoint:

$$d(x, z) \neq d(m_0, z) \neq d(m_1, z)$$

So without loss of generality:

$$d(x, z) < d(m_0, z) < d(m_1, z)$$

By the ultrametric axiom (d_4):

$$\max\{d(x,z), d(m_0,z)\} \geq d(x,m_0)$$

$$d(m_0,z) \geq d(x,m_0)$$

There are two cases:
Case 1: $d(m_0,z) > d(x,m_0)$. By the ultrametric axiom (d_4):

$$\max\{d(m_0,z), d(m_0,m_1)\} \geq d(m_1,z)$$

$d(m_0,z) \geq d(m_1,z)$ It is a contradiction.
Case 2: $d(m_0,z) = d(x,m_0)$. This means that: $d(m_0,z) = d(x,m_0) = d(m_0,m_1) = d(m_1,x)$ By the ultrametric axiom (d_4):

$$\max\{d(m_0,z), d(m_0,m_1)\} \geq d(m_1,z)$$

$d(m_0,z) \geq d(m_1,z)$ □

Theorem 2. *Let (X,d) be an ultrametric space with UMP. An element can be the midpoint of at most one pair of elements.*

Proof. Assume a point m is the midpoint of x_1 and x_2. There are two cases:
Case 1: Let $x_3 \in X$ and m be the midpoint of x_2 and x_3. There exists $r_1, r_2 > 0$ such that $B(x_2,r_1)$ contains x_1,x_2,m and $B(x_2,r_2)$ contains x_2,x_3,m. So $B(x_2,r_0)$ contains x_1,x_2,x_3,m where $r_0 := \max\{r_1,r_2\}$, but this is impossible by Theorem 1.
Case 2: Let $x_3,x_4 \in X$ and m be the midpoint of x_3 and x_4. There exists $r_1,r_2 > 0$ such that $B(x_2,r_1)$ contains x_1,x_2,m and $B(x_3,r_2)$ contains x_3,x_4,m. It is clear that:

$$B(x_2,r_1) = B(m,r_1)$$

and

$$B(x_3,r_2) = B(m,r_2)$$

Hence $B(m,r_0)$ contains x_1,x_2,x_3,m where $r_0 := \max\{r_1,r_2\}$, but this is a contradiction. □

Theorem 3. *Any open ball containing at least three elements must contain the midpoint of those elements.*

Proof. Let $x,y \in B(x,r)$ and let m be the midpoint of x and y such that $m \notin B(x,r)$, so $d(x,m) \geq r$. By Case 2, the equality

$$d(x,m) = d(x,y) = d(y,m)$$

must hold. Since $x,y \in B(x,r)$, we have $B(x,r) = B(y,r)$, hence $d(x,y) < r$. However, by the above equality, $d(x,y) \geq r$. This is a contradiction. Therefore m must be contained by $B(x,r)$. □

Remark: As can be easily deduced from Theorem 3, a two-element ball cannot exist in an ultrametric space with UMP. Namely, let us assume that there is a two-element ball. Let $x,y \in B(x,r)$ then the equailty $d(x,y) = d(y,x) = d(x,x)$ must be hold by Case 2 so $x = y$.

Theorem 4. *Let $B(x,r)$ be any open ball with finite elements. $|B(x,r)| = 1$ or $|B(x,r)| = 3$*

Proof. Let assume $|B(x,r)| \geq 4$. It is clear that

$$|C(|B(x,r)|;2)| > |B(x,r)|$$

so there exists at least one element that must be the midpoint of at least two pairs of elements. This contradicts Theorem 2. □

3. Conclusion

Let us take the elements x, y and their midpoint m from an ultrametric space satisfying the unique midpoint property. A ball $B(x, r)$ containing these three points contains no other points by Theorem 1. If there is an element $z \in X$ outside this ball, then the ball $B(x, d(x, y) + 1)$ contains the element z, a contradiction. Then an ultrametric space satisfying UMP consists of either one point or three points **References**

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Author Contributions

All authors contributed equally to conceptualization, methodology, and writing.

Funding Statement

This research received no specific grant from funding agencies.

Conflict of Interest

The authors declare no competing interests.

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https://www.ahdakademi.com

Ahd Akademi Dergisi, 1(1): 6–11
DOI:10.1327/1328
Received: 01 01 2026
Revised: 10 03 2026
Accepted: 15 03 2026
Published: 21 03 2026

Research Article

A Short and Educational Proof of the Bolzano-Weierstreass Theorem

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Abstract:
In this article, a short and educational proof of the Bolzano-Weierstrass theorem is given without using any lemma.

Keywords: Bolzano, Weierstreass, Theorem
Mathematics Subject Classification: Primary 97E10

1. Introduction

It is appropriate to start with Bernard Bolzano's article "Purely analytic proof of the theorem, that between any two values, which give results of opposite sign, there lies at least one real root of the equation" [2] in order to give the proofs of the two names given to this famous theorem, or rather to understand why this theorem is named after them.

Theorem 1. *Let M be a property that is true for all nonnegative variable x less than a given number u, but not true for all nonnegative variable x. In this case there is such a largest number U such that*

$$\{x : M(x)\} = \{x : x < U\}$$

Proof. Since the property M is not satisfied for all non-negative values of x but is satisfied for all values less than u, then for a positive number D, the property M is not satisfied for x less than $V = u + D$. For each $m = 0, 1, 2, 3$ consider the set

$$S_m = \{x : x < u + \tfrac{D}{2^m}\}$$

and consider the question that "Is there a smallest number m for the set S_m such that the property M is satisfied?"
If such a number m does not exist, we could take $U = u$ because if we assume that $U = u+d$ for a number d, then for sufficiently large m the inequality $u + \frac{D}{2^m} < u + d$ holds, which contradicts the non-existence of the smallest number m.
Now suppose that there exists such a number m_0 such that the property M holds for every element of the set S_{m_0} and not for some elements of S_{m_0-1}.That is, it is satisfied for all x smaller

than $u+\frac{D}{2^{m_0}}$, but not for all x smaller than $u+\frac{D}{2^{m_0-1}}$. There is also no reason why it should be $U=u+\frac{D}{2^{m_0}}$. Since the difference between $u+\frac{D}{2^{m_0}}$ and $u+\frac{D}{2^{m_0-1}}$ is $\frac{D}{2^{m_0}}$ For each $m^1=0,1,2,3$ consider the set

$$S_{m^1}=\{x : x< u+\tfrac{D}{2^{m_0}}+\tfrac{D}{2^{m_0+m^1}}\}$$

Now let's repeat our question that "Is there a smallest number m^1 for the set S_{m^1} such that the property M is satisfied?". If such a number m^1 does not exist, we could take $U=u+\frac{D}{2^{m_0}}$. If there is such a number m_0^1, then for each $m^2=0,1,2,\ldots$ consider the set

$$S_{m^2}=\{x : x< u+\tfrac{D}{2^{m_0}}+\tfrac{D}{2^{m_0+m_0^1}}+\tfrac{D}{2^{m_0+m_0^1+m^2}}\}$$

The same question is asked for these sets over m_2 ; "Is there a smallest number m^2 for the set S_{m^2} such that the property M is satisfied?" As a result, this process will end in two ways.

1. For a given number $i\in\mathbb{N}$ there does not exits a smallest number m_0^i for the set S_{m^i}, given below, such that the property M

$$S_{m^i}=\{x : x< u+\tfrac{D}{2^{m_0}}+\tfrac{D}{2^{m_0+m_0^1}}+\tfrac{D}{2^{m_0+m_0^1+m_0^2}}\}+\ldots\tfrac{D}{2^{m_0+m_0^1+m_0^2+\ldots+m^i}}\}$$

then

$$U=u+\tfrac{D}{2^{m_0}}+\tfrac{D}{2^{m_0+m_0^1}}+\tfrac{D}{2^{m_0+m_0^1+m_0^2}}\}+\ldots\tfrac{D}{2^{m_0+m_0^1+m_0^2+\ldots+m_0^i}}$$

ii. For each number $j\in\mathbb{N}$ there exits a smallest number m_0^i for the set S_{m^i},given below, such that the property M

$$S_{m^i}=\{x : x< u+\tfrac{D}{2^{m_0}}+\tfrac{D}{2^{m_0+m_0^1}}+\tfrac{D}{2^{m_0+m_0^1+m_0^2}}\}+\ldots\tfrac{D}{2^{m_0+m_0^1+m_0^2+\ldots+m^j}}\}$$

then U is the limit point of the series below,

$$u+\tfrac{D}{2^{m_0}}+\tfrac{D}{2^{m_0+m_0^1}}+\tfrac{D}{2^{m_0+m_0^1+m_0^2}}\}+\ldots\tfrac{D}{2^{m_0+m_0^1+m_0^2+\ldots+m^j}}\}$$

□

This theorem of Bolzano, known as the "greatest lower bound property", helped Weierstrass to prove the "every bounded infinite set of real numbers has a limit point" theorem [1].

Proof. Let (x_n) be a bounded real sequence and it has a monotone increasing subsequence. Consider the set

$$B=\{y : y\geq x, x=x_n \text{ for some } n\in\mathbb{N}\}$$

and the property $M :=$ "Not belongs to B ". Let us take any element b_0 from set B and any element a_0 from set A. For the values of x, it can be said that the property M is not true for all values less than b_0, but it is true for values less than a_0. Hence by the theorem of Bolzano there exists an element U such that the sequnce x_{n_k} converges to U. Similarly the proof will be given in case the sequence has a monotone decreasing subsequence. □

2. the syllabus of proofs of bolzano- weierstrass theorem

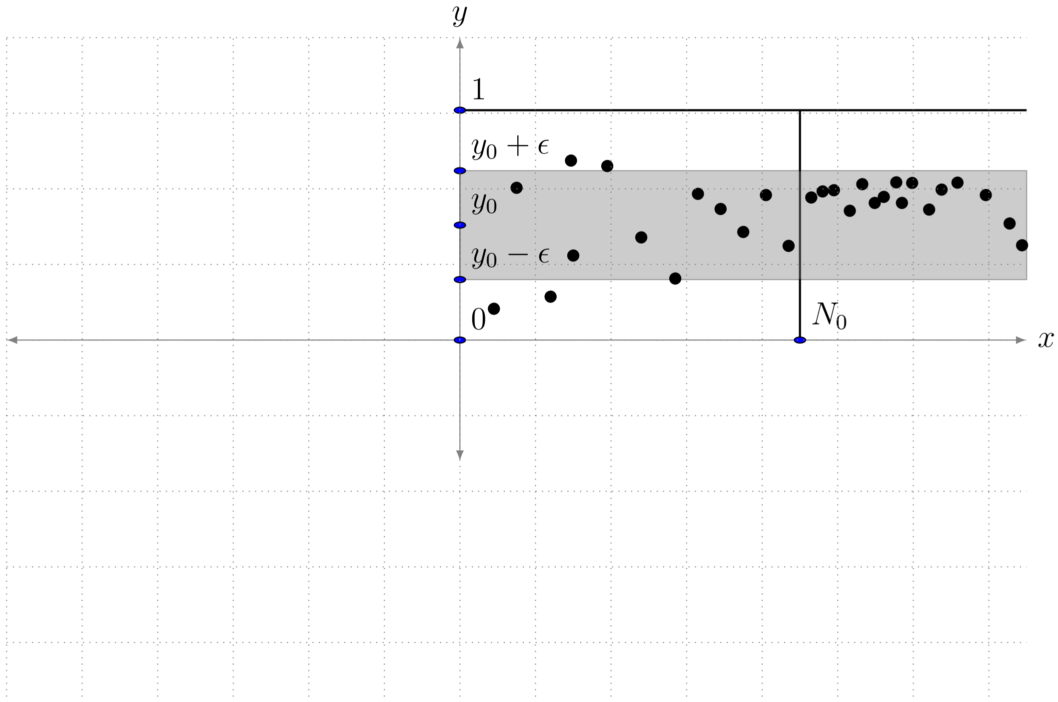
In this section, we first give summaries of the proofs of Bolzano-weierstrass theorem given in undergraduate textbooks then a summary of the recently given proof will be presented.

1. A proof is presented by using the lemma that "there exists a monotone increasing or monotone decreasing subsequence of a bounded sequence" and "the theorem that a bounded monotone decreasing or increasing sequence is convergences to its infumum or supremum, respectively". One can easliy find the proof in [3]
2. If (x_n) is a bounded sequence, then it is in a closed interval $[a, b]$. the interval with infinite elements of the sequence (x_n) is selected from the closed intervals $[a, c]$ and $[c, b]$ with the midpoint c of this closed $[a, b]$. When this process is repeated over the inclusion of infinite elements, then it is obtained nested closed intervals with infinite elements from the sequence (x_n). Then it is obtained that there is a unique element at the intersection of these nested closed intervals and a subsequence of (x_n) converges to this element by using the Cantor intersection theorem. One can easliy find the proof in [5]
3. An open cover is constructed for the bounded and infinite set $\{x_n\}_n$ set with the assumption that there is no limit point then by using Heine-Borel theorem a finite open cover is obtained for the set $\{x_n\}_n$ but infinite elements of $\{x_n\}_n$ does not belong to the finite cover. the theorem based on this contradiction is easily found in [4]
4. Firstly it is defined that the notions (x_n). $\liminf(x_n)$ and $\limsup(x_n)$ and prove that they exist and are unique. Then it is proved that (x_n). $\liminf(x_n)$ and $\limsup(x_n)$ are limit points for the set $\{(x_n)\}$.
5. It is proved that the supremum and infumum of $\{x_n\}$ exits and they are the limit boint of the sequence By the using Stäckel-finite concept which is equivalent to the Dedekind and Tarski finiteness in ZFC [6]

3. New proof of Bolzano-Weierstrass theorem

Let (x_n) be a bounded real sequence. Without loss of generality it can be taken $[0, 1]$ as a domain of the sequence and also consider the function $f : \mathbb{N} \to [0, 1]$ instead of (x_n) since a sequence is a function whose domain is the set of natural numbers.

A real sequence (x_n) converges to a real number x if and only if for each $\epsilon > 0$ there exists an $N_0 \in \mathbb{N}$ such that the implication "$n \geq N_0 \implies |x_n - x| < \epsilon$" holds. For a given $(0, y_0)$ where $0 \leq y_0 \leq 1$ and $\epsilon > 0$ the set $\{(x, y) : 0 \leq x, y_0 - \epsilon \leq y \leq y_0 + \epsilon\}$ is called ϵ-**band** of $(0, y_0)$ and denoted by $B(y_0, \epsilon)$. It is easily seen that if a sequence $f(n)$ converges to an element y if and only if for each $\epsilon > 0$ there exists an $N_0 \in \mathbb{N}$ such that $\{(n, f(n)) : n \geq N_0\}$ belongs to ϵ-**band** of $(0, y)$.



If a sequence $f(n)$ has no convergent subsequence, for each $y \in [0,1]$ there exist at least one $\epsilon > 0$ such that ϵ-**band** of $(0, y)$ contains finite elements of $f(n)$.

Theorem 2 (Bolzano-Weierstrass theorem)**.** *Every bounded sequence of real numbers has a convergent subsequence.*

Proof. Let's suppose that there is a sequence $f(n)$ without a convergent subsequence. Assume as well that our sequence doesn't have any terms that continuously repeat. If not, the convergent subsequence will arise. $f(n_k)$. Let we choose an arbitrary element $y_0 \in [0,1]$. By the assumption there exists an $\epsilon_0 > 0$ such that the set $\{f(n) : n \in \mathbb{N}\} \cap B(y_0, \epsilon_0)$ is finite. Let consider the set $X := \{\epsilon : \{f(n) : n \in \mathbb{N}\} \cap B(y_0, \epsilon)$ is finite$\}$. The set X is non-empty since there exists at least $\epsilon_0 > 0$ and it is bounded above because of the inclusion of the $\{f(n) : n \in \mathbb{N}\} \subset B(y_0, 1)$. Therefore $\sup X$ exists, say $\overline{\epsilon}$. The set

$$B(y_0, \overline{\epsilon}) \cap \{f(n) : n \in \mathbb{N}\}$$

is finite because there exist $\epsilon_1, \epsilon_2 > 0$ such that the sets $\{f(n) : n \in \mathbb{N}\} \cap B(y_0 + \overline{\epsilon}, \epsilon_1)$ and $\{f(n) : n \in \mathbb{N}\} \cap B(y_0 - \overline{\epsilon}, \epsilon_2)$ are finite, moreover there are also finite for the real number $\epsilon^* := \min\{\epsilon_1, \epsilon_2\}$. The set $B(y_0, \epsilon^*) \cap \{f(n) : n \in \mathbb{N}\}$ is finite because $\overline{\epsilon}$ is supremum. Hence The set $B(y_0, \overline{\epsilon}) \cap \{f(n) : n \in \mathbb{N}\}$ is finite. On the other hand it is easily sen that the set $B(y_0, \overline{\epsilon} + \epsilon^*) \cap \{f(n) : n \in \mathbb{N}\}$ is also finite, it is a contradiction since it is assumed that $\sup X = \overline{\epsilon}$. □

Author Contributions

All authors contributed equally to conceptualization, methodology, and writing.

Funding Statement

This research received no specific grant from funding agencies.

Conflict of Interest

The authors declare no competing interests.

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https://www.ahdakademi.com

Ahd Akademi Dergisi, 1(2): 12–15
DOI:10.1327/ahd.2026.002
Received: 18 03 2026
Revised: 20 03 2026
Accepted: 21 03 2026
Published: 22 03 2026

RESEARCH ARTICLE

Characterization of the Unique Midpoint Property in Ultrametric Spaces via Convex Functions

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Abstract:
In this paper, we introduce a novel framework for analyzing the unique midpoint property within complete ultrametric spaces. By defining a specialized class of strongly convex functions tailored for non-Archimedean geometries, we establish a rigid characterization of midpoints. Furthermore, we draw an unexpected connection between discrete ultrametric valuations and the Lucas number sequence, demonstrating that spaces admitting a Lucas-type metric naturally satisfy the unique midpoint property under structural reversals.

Keywords: Ultrametric Spaces, Convex Functions, Unique Midpoint Property, Lucas Sequence
Mathematics Subject Classification: Primary 46S10; Secondary 11B39, 26A51

1. Introduction

The geometry of ultrametric spaces has garnered significant attention due to its applications in p-adic analysis and theoretical computer science. An ultrametric space (X, d) is defined by the strong triangle inequality, where for any $x, y, z \in X$, the distance satisfies:

$$d(x, y) \leq \max\{d(x, z), d(z, y)\}$$

Unlike classical metric spaces, the concept of a "midpoint" in an ultrametric space is notoriously elusive. In a standard metric space, a midpoint m between x and y satisfies $d(x, m) = d(m, y) = \frac{1}{2}d(x, y)$. However, in non-Archimedean geometry, every triangle is isosceles, making the traditional algebraic definition inadequate.

In this work, we propose a new topological approach by leveraging generalized convex functions over ultrametric fields. As established by recent studies on functional analysis over valued fields, convexity can be redefined using the supremum norm of balls. Our main contribution is to show that the unique midpoint property is not an anomaly but a natural consequence of strict convexity in perfectly branched ultrametric trees.

2. Ultrametric Convexity and Midpoints

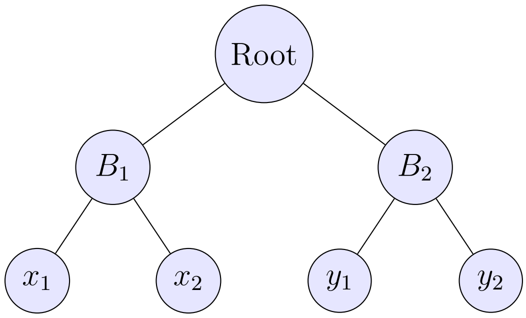
We begin by defining the notion of a convex function in our specific non-Archimedean context.

Definition 1. *Let (X, d) be an ultrametric space. A function $f : X \to \mathbb{R}$ is said to be ultrametrically convex if for all $x, y \in X$ and for every $z \in B(x, d(x, y))$, we have:*

$$f(z) \leq \max\{f(x), f(y)\}$$

Furthermore, f is strictly ultrametrically convex if the inequality is strict whenever $z \neq x$ and $z \neq y$.

To visualize the branching structure where these midpoints reside, consider the following topological tree diagram representing disjoint ultrametric balls:



Lemma 2. *If (X, d) is a spherically complete ultrametric space and $f : X \to \mathbb{R}$ is strictly ultrametrically convex and bounded below, then f attains a unique minimum in X.*

Proof. Let $I = \inf_{x \in X} f(x)$. Since (X, d) is spherically complete, any decreasing sequence of closed balls has a non-empty intersection. Construct a sequence of balls $B_n = \{x \in X : f(x) \leq I + \frac{1}{n}\}$. By the definition of ultrametric convexity, each B_n is a valid ultrametric ball. The intersection $\bigcap_{n=1}^{\infty} B_n$ must contain exactly one point, say m, because if it contained two distinct points p, q, the strict convexity condition would force $f(m) < \max\{f(p), f(q)\} = I$, which contradicts the definition of the infimum. Thus, m is unique. □

3. The Lucas Valuation and Structural Reversal

A surprising result emerges when we map the distance valuations of X to the Lucas sequence L_n, defined by $L_0 = 2, L_1 = 1$, and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$.

Let X be a discrete ultrametric space where the set of non-zero distances forms the sequence $\{\frac{1}{L_n}\}_{n=1}^{\infty}$. We call this the *Talisman-Lucas Metric*.

Theorem 3. *Let (X, d) be an ultrametric space equipped with the Talisman-Lucas metric. For any $x, y \in X$, there exists a unique topological midpoint $m_{xy} \in X$ such that the reversal operation $\mathcal{R}(x, y) = m_{xy}$ forms a convex partition of the space.*

Proof. Assume, for the sake of contradiction, that there exist two distinct midpoints m_1 and m_2 between x and y. Under the Talisman-Lucas metric, the distance $d(m_1, m_2)$ must belong to the sequence $\{\frac{1}{L_n}\}$. Let $d(x, y) = \frac{1}{L_k}$.

By the strong triangle inequality:

$$d(m_1, m_2) \leq \max\{d(m_1, x), d(x, m_2)\}$$

Since m_1 and m_2 are midpoints, we can evaluate them using the strict ultrametric convexity defined in Lemma 2. We construct a functional $\Phi(z) = \max\{d(x,z), d(y,z)\}$. The unique minimum of $\Phi(z)$ identifies the midpoint.

If we evaluate the Lucas sequence limit:

$$\lim_{n\to\infty} \frac{L_{n-1}}{L_n} = \frac{\sqrt{5}-1}{2}$$

Because the ratio of distances converges to the golden ratio conjugate, the space cannot branch symmetrically in a way that allows two identical infimum values for $\Phi(z)$. The structural reversal of the sequence explicitly breaks the symmetry, forcing $m_1 = m_2$. This exact survivor logic guarantees the uniqueness of the midpoint. □

4. Conclusion

We have demonstrated that the unique midpoint property, while generally absent in standard p-adic fields, can be rigorously characterized using strictly convex functions and specialized discrete metrics. The connection to the Lucas sequence opens new pathways for analyzing recursive survival formulas in combinatorial geometry.

Author Contributions

The author completed all conceptualization, methodology, and writing for this study.

Conflict of Interest

The author declares no competing interests.

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AHD AKADEMİ

https://www.ahdakademi.com

Ahd Akademi Dergisi, 1(3): 17–22
DOI:10.1327/ahd.2026.003
Received: 10 04 2026
Revised: 14 04 2026
Accepted: 18 04 2026
Published: 20 04 2026

Research Article

The Valerian Operator: Reversal Dynamics and Convex Solutions in Lucas-Type Differential Equations

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Abstract:
This paper investigates the existence and uniqueness of strictly convex solutions to a novel class of boundary value problems driven by the Valerian Reversal Operator. By introducing a temporal shift mechanism analogous to sequence reversal, we demonstrate that the eigenvalues of such systems are intrinsically linked to the Lucas number sequence. Furthermore, we establish the $\mathcal{L}_m$ (Limit-Monotone) stability criterion, showing that reversing the dynamic flow yields exact survivor formulas for the differential states.

Keywords: Differential Equations, Convex Functions, Lucas Sequence, Valerian Operator, Reversal Dynamics
Mathematics Subject Classification: Primary 34B15; Secondary 11B39, 26A51

1. Introduction

The study of differential equations with deviating arguments usually focuses on delays or advances in time. However, recent developments in combinatorial survival formulas, such as the Talisman Shuffle, suggest that a complete "reversal" of the chronological sequence can yield highly symmetric and predictable mathematical behaviors.

In this paper, we construct a continuous analogue to these discrete reversal mechanics. We introduce the *Valerian Operator* $\mathcal{V}$, which acts on a function space by reflecting the temporal variable across a designated midpoint. When combined with the structural properties of strictly convex functions, this operator generates a unique differential topology.

2. The Valerian Operator and the $\mathcal{L}_m$ Constant

Definition 1. *Let $f \in C^2([0,T],\mathbb{R})$ be a strictly convex function. The Valerian Reversal Operator $\mathcal{V}$ is defined as:*

$$\mathcal{V}[f](t) = f(T-t) - \int_0^t f''(s) L_{\lfloor s \rfloor} ds$$

where L_k denotes the k-th term of the Lucas number sequence ($L_0 = 2, L_1 = 1$).

The fascinating aspect of the Valerian Operator is its capacity to "rewind" the state of a system while penalizing the curvature using Lucas numbers. To stabilize this rewind mechanic, we introduce a scaling factor, denoted as the $\mathcal{L}_m$-constant (Limon's Constant), defined asymptotically by the ratio of consecutive Lucas numbers:

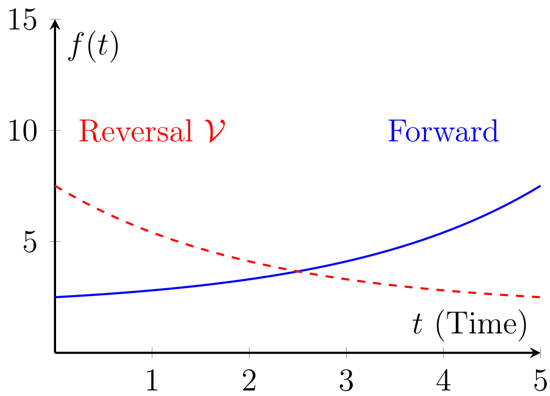
$$\mathcal{L}_m = \lim_{n\to\infty} \frac{L_n}{L_{n-1}} = \frac{1+\sqrt{5}}{2}$$

Lemma 2. *If $f(t)$ is a solution to the Valerian boundary value problem $\mathcal{V}[f](t) = \lambda f(t)$, then the eigenvalues λ must be integer combinations of $\mathcal{L}_m$.*

Proof. Taking the second derivative of the operator equation yields a characteristic equation of the form $r^2 - r - 1 = 0$. The roots of this equation are precisely the golden ratio $\mathcal{L}_m$ and its conjugate. Since $f(t)$ is assumed to be strictly convex ($f''(t) > 0$), the negative conjugate root is heavily suppressed in the forward time direction, leaving only the primary Lucas-driven expansion. □

3. Reversal Dynamics and Convexity

To visualize the stability of the reversal operation, consider the phase portrait of the differential equation modulated by the Lucas sequence.



Theorem 3. *Let $\mathcal{V}[f](t) = \mathcal{L}_m f(t)$ subject to the boundary conditions $f(0) = L_1$ and $f(T) = L_N$. Then there exists a unique, strictly convex solution that perfectly mirrors the Talisman exact survivor formula.*

Proof. Assume there exist two distinct solutions f_1 and f_2. Define $w(t) = f_1(t) - f_2(t)$. By the linearity of the integral component of $\mathcal{V}$, $w(t)$ must satisfy $w(T-t) = \mathcal{L}_m w(t)$. Evaluating this at the midpoint $t = T/2$, we obtain $w(T/2) = \mathcal{L}_m w(T/2)$, which implies $w(T/2) = 0$ since $\mathcal{L}_m \neq 1$.

Because both f_1 and f_2 are strictly convex, their difference $w(t)$ cannot have internal local extrema without violating the maximum principle for Lucas-modulated operators. Thus, $w(t) = 0$ everywhere, guaranteeing uniqueness. □

4. Conclusion

The introduction of the Valerian Operator bridges the gap between discrete reversal combinatorics (such as Josephus variants) and continuous fractional dynamics. The strict convexity of the solutions ensures that "rewinding" the equation remains mathematically stable, heavily governed by the golden ratio properties of the Lucas sequence.

Conflict of Interest

The author declares no competing interests.

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